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From p -branes to Cosmology

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ABSTRACT

We study the relationship between static p -brane solitons and cosmological solutions of string theory or M-theory. We discuss two different ways in which extremal p -branes can be generalised to non-extremal ones, and show how wide classes of recently discussed cosmological models can be mapped into non-extremal p -brane solutions of one of these two kinds. We also extend previous discussions of cosmological solutions to include some that make use of cosmological-type terms in the effective action that can arise from the generalised dimensional reduction of string theory or M-theory.

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1 Introduction

A considerable literature exists on the subject of p -brane solutions of the low-energy limits of string theories and M-theory. These include the BPS-saturated classes of extremal p -branes, which preserve some fraction of the spacetime supersymmetry, and also their non-extremal generalisations, where supersymmetry is completely broken. A characteristic feature of all these solitonic solutions is that they are static (or at least stationary, if one allows rotating solutions as well). Other studies have focussed on a different class of solutions to the low-energy effective equations of motion, namely cosmological solutions in which the metric evolves in time. Typically, these solutions describe the evolution of a universe from a small initial radius to a large radius at late times.

Although they are ostensibly very different in their interpretation, it has been observed that there are actually many similarities between some of the p -brane solutions and the cosmological solutions [1]. Indeed in some examples, an exact correspondence has been established [2]. The most obvious difference between the two classes of solution is that the fields of the cosmological solutions depend upon time, while in the p -brane solitons they depend instead on the coordinates of the transverse space (in fact, in the cases of relevance here, they depend just on the radial distance from the origin in the transverse space). However, this distinction is not such a profound one; it is, for example, well known that in the interior region of the Schwarzschild black hole the usual time coordinate becomes spacelike, while the usual radial coordinate acquires a new interpretation as the time coordinate [3]. Thus one way, at least, in which an equivalence can arise is that the interior solution of a black p -brane may be re-interpreted as a cosmological model. As we shall see later, not all of the mappings between p -brane and cosmological solutions are of this kind, and sometimes it is necessary to make Wick rotations in order to interchange the timelike and spacelike character of coordinates in the two cases.

The purpose of this paper is to investigate the mapping between p -brane solutions and cosmological solutions in a somewhat general framework. In order to do so, we shall begin by first obtaining a large class of cosmological solutions. In general, our starting point can be taken to be a theory describing the coupling of gravity to a dilatonic scalar field and an antisymmetric tensor field strength of degree n . A special case of this, which has not been extensively discussed in the literature, is when the degree n is actually zero. In this case the “field strength” is really just a constant; if there were no coupling to a dilaton, it would in fact correspond precisely to a pure cosmological constant term. In section 2, we begin by setting up appropriate ansätze for the metric and field strength that enable us to study

rather broad classes of cosmological solutions, including in particular the ones utilising a 0-form field strength, which we examine in section 3.

In section 4, we examine the relationship between the cosmological solutions obtained in the previous sections, and non-extremal p -brane solitons. In order to do this, it is necessary to distinguish between two different kinds of generalisation of the standard BPS-saturated extremal p -branes. In the first of these, which we shall refer to as type 1 non-extremal p -branes, the form of the D -dimensional metric remains the same as in the extremal case, namely $ds^2 = e^{2A} dx^\mu dx_\mu + e^{2B} (dr^2 + r^2 d\Omega^2)$, where A and B are functions of r . However, whereas in the extremal case the function $X \equiv dA + \tilde{d}B$ is equal to zero, here in the non-extremal generalisation it becomes non-vanishing [4]. (We have defined $d = p + 1$ and $\tilde{d} = D - d - 2$.) The other kind of non-extremal generalisation, which we shall refer to as type 2, begins from a modified form for the metric, namely $ds^2 = e^{2A} (-e^{2f} dt^2 + dx^i dx^i) + e^{2B} (e^{-2f} dr^2 + r^2 d\Omega^2)$ [5, 6]. In this case, the relation $dA + \tilde{d}B = 0$ is still maintained. Although both the type 1 and type 2 non-extremal generalisations introduce an additional function, namely X or f , the way in which they enter the metric ansatz is quite different, although the two become equivalent when $p = 0$. As we shall see, both types of non-extremal p -brane generalisation can be mapped over into cosmological solutions. The more standard cosmological solutions actually correspond to the type 1 black p -branes; for this reason, we shall refer to such cosmological solutions as type 1. These are the kind that are constructed in sections 2 and 3. We also discuss some of the cosmological solutions associated with the type 2 black p -branes; some examples of this type have been considered in [2]. Finally in section 4, we also show how certain special cases of the type 1 solutions can in fact be obtained by a dimensional reduction of type 2 solutions, where the original time coordinate is used for the compactification process.

2 General cosmological solutions in string or M theories

The study of the cosmological consequences of string theory has been an area of much active research in the past [7-11]. Recently, considerable attention has been directed to string-inspired cosmology in various dimensions [12-22]. Cosmological models are described by solutions of the low-energy effective theory in which the metric tensor, and the other fields, are time dependent. We shall consider cosmological metrics of the form

$$ds^2 = -e^{2U} dt^2 + e^{2A} d\bar{s}_q^2 + e^{2B} d\bar{s}_q^2, \quad (2.1)$$

where U , A and B are functions only of t , and $d\bar{s}_q^2$ and $d\bar{s}_{\tilde{q}}^2$ denote metrics on maximally-symmetric spaces of positive, negative or zero curvature, with dimensions q and \tilde{q} respectively, *i.e.*

$$d\bar{s}_q^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2, \quad d\bar{s}_{\tilde{q}}^2 = \frac{d\tilde{r}^2}{1 - \tilde{k}\tilde{r}^2} + \tilde{r}^2 d\tilde{\Omega}^2, \quad (2.2)$$

where $d\Omega^2$ and $d\tilde{\Omega}^2$ are the metrics on unit $(q-1)$ - and $(\tilde{q}-1)$ -spheres respectively. For later convenience, we define $d = q-1$ and $\tilde{d} = \tilde{q}-1$ and hence $d + \tilde{d} = D-2$. The constant k and \tilde{k} can be taken to be equal to 0, 1, or -1 independently, corresponding to flat, spherical or hyperboloidal spatial sections respectively. A large class of cosmological solutions with $\tilde{k} = 0$ were obtained in [19]. The solutions include some realistic models, where the $(q+1)$ -dimensional spacetime expands at large time, while the \tilde{q} -dimensional space contracts and becomes unobservable at large time. Note that when $\tilde{k} = 0$, the solution can be compactified on any \tilde{q} -dimensional Ricci-flat space with metric $d\bar{s}_{\tilde{q}}^2$. In this section, we generalise the previously-known solutions to include the cases where $\tilde{k} \neq 0$, and briefly discuss their cosmological characteristics.

In the vielbein basis $e^0 = e^U dt$, $e^a = e^A \bar{e}^a$, $e^\alpha = e^B \bar{e}^\alpha$, we find that the curvature 2-forms are given by

$$\begin{aligned} \Theta^0_a &= e^{-2U} (\ddot{A} - \dot{U}\dot{A} + \dot{A}^2) e^0 \wedge e^a, \\ \Theta^0_\alpha &= e^{-2U} (\ddot{B} - \dot{U}\dot{B} + \dot{B}^2) e^0 \wedge e^\alpha, \\ \Theta^a_b &= \bar{\Theta}^a_b + e^{-2U} \dot{A}^2 e^a \wedge e^b, \\ \Theta^\alpha_\beta &= \bar{\Theta}^\alpha_\beta + e^{-2U} \dot{B}^2 e^\alpha \wedge e^\beta, \\ \Theta^a_\beta &= e^{-2U} \dot{A}\dot{B} e^a \wedge e^\beta, \end{aligned} \quad (2.3)$$

where a dot denotes a derivative with respect to the time coordinate t , $\bar{\Theta}^a_b$ is the curvature 2-form for the metric $d\bar{s}_q^2 = \bar{e}^a \bar{e}^a$ in the vielbein basis \bar{e}^a , and $\bar{\Theta}^\alpha_\beta$ is the curvature 2-form for the metric $d\bar{s}_{\tilde{q}}^2 = \bar{e}^\alpha \bar{e}^\alpha$ in the vielbein basis \bar{e}^α . It follows that the tangent-space components of the Ricci tensor for the metric (2.1) are given by

$$\begin{aligned} R_{00} &= -e^{-2U} (q(\ddot{A} + \dot{A}^2 - \dot{U}\dot{A}) + \tilde{q}(\ddot{B} + \dot{B}^2 - \dot{U}\dot{B})), \\ R_{ab} &= e^{-2U} (\ddot{A} + q\dot{A}^2 - \dot{U}\dot{A} + \tilde{q}\dot{A}\dot{B}) \delta_{ab} + e^{-2A} \bar{R}_{ab}, \\ R_{\alpha\beta} &= e^{-2U} (\ddot{B} + \tilde{q}\dot{B}^2 - \dot{U}\dot{B} + q\dot{A}\dot{B}) \delta_{\alpha\beta} + e^{-2B} \bar{R}_{\alpha\beta}. \end{aligned} \quad (2.4)$$

where \bar{R}_{ab} and $\bar{R}_{\alpha\beta}$ denote the tangent-space components of the Ricci tensor for the q -dimensional and \tilde{q} -dimensional spatial metrics. In all the cases we shall consider, these metrics will be Einstein, and we may write $\bar{R}_{ab} = k(q-1)\delta_{ab}$ and $\bar{R}_{\alpha\beta} = \tilde{k}(\tilde{q}-1)\delta_{\alpha\beta}$.

We begin by solving for the simplest cosmological solutions in D dimensions, which involve only the metric, a dilaton and an n -rank antisymmetric field strength F_n . The Lagrangian is given by

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2n!}e^{a\phi}F_n^2, \quad (2.5)$$

where the constant a can be conveniently parameterised as

$$a^2 = \Delta - \frac{2(n-1)(D-n-1)}{D-2}, \quad (2.6)$$

and the constant Δ is preserved under the Kaluza-Klein dimensional reduction [23]. In supergravities, the full bosonic Lagrangian can be consistently truncated to the single-scalar Lagrangian (2.5) for $\Delta = 4/N$, where N is a set of integers $1, 2, \dots, N_{\max}$, and N_{\max} depends on D and n [24]. For 1-form field strengths, additional values of Δ can arise, namely $\Delta = 24/(N(N+1)(N+2))$ [25]. The equations of motion from the Lagrangian (2.5) are

$$\begin{aligned} \square\phi &= \frac{a}{2n!}e^{a\phi}F^2, & \partial_{M_1}(e^{a\phi}F^{M_1\cdots M_n}) &= 0, \\ R_{MN} &= \frac{1}{2}\partial_M\phi\partial_N\phi + S_{MN}, \end{aligned} \quad (2.7)$$

where S_{MN} is a symmetric tensor given by

$$S_{MN} = \frac{1}{2(n-1)!}e^{a\phi}\left(F_{MN}^2 - \frac{n-1}{n(D-1)}F^2g_{MN}\right), \quad (2.8)$$

There are two types of ansätze for the field strength F_n that are compatible with the symmetries of the metric (2.1), giving rise to “electric” and “magnetic” cosmological solutions. In the electric solutions, the ansatz for the antisymmetric tensor is given in terms of its potential, and in a coordinate frame takes the form

$$A_{\alpha_1\alpha_2\cdots\alpha_{\tilde{q}}} = f\epsilon_{\alpha_1\alpha_2\cdots\alpha_{\tilde{q}}}, \quad (2.9)$$

and hence

$$F_{0\alpha_1\alpha_2\cdots\alpha_{\tilde{q}}} = \dot{f}\epsilon_{\alpha_1\alpha_2\cdots\alpha_{\tilde{q}}}, \quad (2.10)$$

where f is a function of t only. Here and throughout this paper $\epsilon_{M\cdots N}$ and $\epsilon^{M\cdots N}$ are taken to be tensor densities of weights -1 and 1 respectively, with purely numerical components ± 1 or 0 . Note in particular that they are not related just by raising and lowering indices using the metric tensor. For electric solutions, we have $q = D - n$ and $\tilde{q} = n - 1$.

For the magnetic cosmological solutions, the ansatz for the tangent-space components for the antisymmetric tensor is

$$F_{a_1a_2\cdots a_q} = \lambda e^{-qA}\epsilon_{a_1a_2\cdots a_q}, \quad (2.11)$$

where λ is a constant. Thus we have $q = n$ and $\tilde{q} = D - n - 1$. The form of the exponential prefactor is determined by the requirement that F_n satisfy the Bianchi identity $dF_n = 0$.

Substituting the ansätze for the metric and the field strength into the equations of motion (2.7), we find

$$\begin{aligned}
\ddot{\phi} + (q\dot{A} + \tilde{q}\dot{B} - \dot{U})\dot{\phi} &= \frac{1}{2}\epsilon a \lambda^2 e^{-\epsilon a \phi - 2qA + 2U} , \\
\ddot{A} + (q\dot{A} + \tilde{q}\dot{B} - \dot{U})\dot{A} + k(q-1)e^{2U-2A} &= \frac{\tilde{q}}{2(D-2)}\lambda^2 e^{-\epsilon a \phi - 2qA + 2U} , \\
\ddot{B} + (q\dot{A} + \tilde{q}\dot{B} - \dot{U})\dot{B} + \tilde{k}(\tilde{q}-1)e^{2U-2B} &= -\frac{q-1}{2(D-2)}\lambda^2 e^{-\epsilon a \phi - 2qA + 2U} , \\
q(\ddot{A} + \dot{A}^2 - \dot{U}\dot{A}) + \tilde{q}(\ddot{B} + \dot{B}^2 - \dot{U}\dot{B}) + \frac{1}{2}\dot{\phi}^2 &= -\frac{q-1}{2(D-2)}\lambda^2 e^{-\epsilon a \phi - 2qA + 2U} ,
\end{aligned} \tag{2.12}$$

where $\epsilon = 1$ for the electric case and $\epsilon = -1$ for the magnetic case. In the electric case, the constant λ arises as the integration constant for the function f in (2.9).

Let us first consider the equations of motion (2.12) for single-scalar cosmological solutions in D dimensions. It is convenient to make the gauge choice $U = qA + \tilde{q}B$, and to define

$$Z_1 = A - \frac{\tilde{q}}{\epsilon a(D-2)}\phi , \quad Z_2 = B + \frac{q-1}{\epsilon a(D-2)}\phi . \tag{2.13}$$

The equations of motion for Z_1 , Z_2 and ϕ become

$$\begin{aligned}
\ddot{Z}_1 &= -k(q-1)e^{2(q-1)Z_1 + 2\tilde{q}Z_2} , \\
\ddot{Z}_2 &= -\tilde{k}(\tilde{q}-1)e^{2qZ_1 + 2(\tilde{q}-1)Z_2 + 2\epsilon\phi/a} , \\
\ddot{\phi} &= \frac{1}{2}\epsilon a \lambda^2 e^{2\tilde{q}Z_2 - \epsilon\Delta\phi/a} ,
\end{aligned} \tag{2.14}$$

together with the first integral:

$$\begin{aligned}
(2\tilde{q}\dot{Z}_2 - \frac{\Delta}{\epsilon a}\dot{\phi})^2 + \Delta\lambda^2 e^{2\tilde{q}Z_2 - \epsilon\Delta\phi/a} + \frac{2\tilde{q}(D-2)a^2}{q-1}\dot{Z}_2^2 \\
= \frac{2q\Delta}{q-1}((q-1)\dot{Z}_1 + \tilde{q}\dot{Z}_2)^2 + 2\Delta k q(q-1)e^{2(q-1)Z_1 + 2\tilde{q}Z_2} \\
+ 2\Delta\tilde{k}\tilde{q}(\tilde{q}-1)e^{2qZ_1 + 2(\tilde{q}-1)Z_2 + 2\phi/(\epsilon a)} .
\end{aligned} \tag{2.15}$$

For generic values of k , \tilde{k} and the charge parameter λ , the equations of motion (2.14) cannot be solved in a closed form. The following are special cases where solutions can be obtained:

Case 1:

A special class of solutions were obtained in [19], namely those with $\tilde{k} = 0$, but with k arbitrary. In this case, a further redefinition of fields was performed, namely

$$X = \tilde{q}Z_2 + (q-1)Z_1 , \quad Y = Z_2 , \quad \Phi = -\frac{\Delta}{\epsilon a}\phi - 2\tilde{q}Z_2 . \tag{2.16}$$

These fields satisfy the following equations:

$$\ddot{X} + k(q-1)^2 e^{2X} = 0, \quad \ddot{\Phi} + \frac{1}{2}\Delta\lambda^2 e^{\Phi} = 0, \quad \ddot{Y} = 0. \quad (2.17)$$

It follows from (2.15) that the first integral corresponds to a conserved Hamiltonian for the above equations,

$$\dot{\Phi}^2 + \Delta\lambda^2 e^{\Phi} + \frac{2\tilde{q}(D-2)a^2}{q-1}\dot{Y}^2 = \frac{2q\Delta}{q-1}(\dot{X}^2 + k(q-1)^2 e^{2X}). \quad (2.18)$$

Thus X and Φ both satisfy Liouville equations. The manifest positivity of the left-hand side of (2.18) (assuming $\Delta > 0$) shows that the Hamiltonian $\dot{X}^2 + k(q-1)^2 e^{2X}$ for X must be positive, and hence the appropriate form of the solution is

$$\begin{aligned} e^{-X} &= \begin{cases} \frac{q-1}{c} \cosh(ct + \delta), & \text{if } k = 1; \\ \frac{q-1}{c} \sinh(ct + \delta), & \text{if } k = -1; \end{cases} \\ X &= -ct - \delta, \quad \text{if } k = 0, \end{aligned} \quad (2.19)$$

where c and δ are constants. Note that in taking the square root of e^{2X} , the positive root should be chosen in the expression for e^{-X} . The Hamiltonian $\dot{\Phi}^2 + \Delta\lambda^2 e^{\Phi}$ for Φ is also manifestly positive, and so the solution can be written as

$$e^{-\frac{1}{2}\Phi} = \frac{\lambda\sqrt{\Delta}}{2\beta} \cosh(\beta t + \gamma), \quad (2.20)$$

where β and γ are constants. The solution for Y may be taken to be simply

$$Y = -\mu t. \quad (2.21)$$

The constraint (2.18) therefore implies that

$$\beta^2 = \frac{q\Delta c^2 - \tilde{q}(D-2)a^2\mu^2}{2(q-1)}. \quad (2.22)$$

In terms of the original functions A , B and U appearing in the metric (2.1), and the dilaton ϕ , the solution takes the form

$$\begin{aligned} e^{\frac{\Delta(D-2)}{2\tilde{q}}A} &= \frac{\lambda\sqrt{\Delta}}{2\beta} \cosh(\beta t + \gamma) e^{\frac{a^2(D-2)\mu t}{2(q-1)}} e^{\frac{\Delta(D-2)}{2\tilde{q}(q-1)}X}, \\ e^{-\frac{\Delta(D-2)}{2(q-1)}B} &= \frac{\lambda\sqrt{\Delta}}{2\beta} \cosh(\beta t + \gamma) e^{\frac{a^2(D-2)\mu t}{2(q-1)}}, \\ e^{\frac{\Delta}{2\epsilon a}\phi} &= \frac{\lambda\sqrt{\Delta}}{2\beta} \cosh(\beta t + \gamma) e^{-\mu q t}, \end{aligned} \quad (2.23)$$

together with $U = qA + \tilde{q}B$. In a case where there is no dilaton, the solutions for A and B are again given by (2.23), with $\mu = 0$. If instead $\tilde{q} = 0$, we have from (2.6) that $a^2 = \Delta$;

the solution for ϕ follows from (2.23) by setting $\tilde{q} = 0$, and A is given by $A = X/(q - 1)$, with X given by (2.19).

The solutions presented above are single-scalar solutions, involving a single field strength carrying an electric or a magnetic charge. In fact, when $\tilde{k} = 0$ one can also obtain general solutions for multi-scalar, or dyonic cosmological models [19]. We shall see in section 4 that these solutions are closely related to static p -brane solutions.

Case 2:

In this case we have $\tilde{k} \neq 0$, but with $k = 0$. The solution for the function Z_1 is easily found to be $Z_1 = \mu t$. To simplify the equations of motion for Z_2 and ϕ , we define

$$u_1 = Z_2 + \frac{q\Delta}{b}Z_1, \quad u_2 = -\frac{\phi}{\epsilon a} - \frac{2q\tilde{q}}{b}Z_1, \quad (2.24)$$

which satisfy

$$\ddot{u}_1 = -\tilde{k}(\tilde{q} - 1)e^{2(\tilde{q}-1)u_1-2u_2}, \quad \ddot{u}_2 = -\frac{1}{2}\lambda^2 e^{2\tilde{q}u_1+\Delta u_2}, \quad (2.25)$$

where $b = (\Delta + 2)\tilde{q} - \Delta$. If instead $\tilde{k} = 0$, the equations can be solved in general, reducing to the situation discussed in case 1. If both \tilde{k} and the charge parameter λ are non-vanishing, we cannot obtain the general solution to these equations. However, there exists a special solution where \dot{u}_1 and \dot{u}_2 are proportional to one another. Thus we may make the ansatz $u_2 = c_1 u_1 + c$, where c and c_1 are constants. The two equations (2.25) reduce to a single Liouville equation, if $c_1 = -2/(\Delta + 2)$ and $e^{(\Delta+2)c} = -4\tilde{k}(\tilde{q} - 1)/(\lambda^2(\Delta + 2))$. Defining $u = b/(\Delta + 2)u_1$, this Liouville equation is

$$\ddot{u} = \frac{1}{4}b\lambda^2 e^{\Delta c} e^{2u}. \quad (2.26)$$

The first order equation (2.15) becomes

$$\dot{u}^2 - \frac{1}{4}b\lambda^2 e^{2u+\Delta c} = \frac{q(D-2)a^2\mu^2}{2 + (\Delta + 2)\tilde{q}}. \quad (2.27)$$

The Liouville equation (2.26) with the first-order constraint (2.27) can be easily solved, giving finally

$$e^{-u} = \frac{\lambda\sqrt{b}}{2\beta} e^{\frac{\Delta}{2}c} \sinh(\beta t + \alpha), \quad (2.28)$$

$$Z_1 = \mu t, \quad u_1 = \frac{\Delta + 2}{b}u, \quad u_2 = -\frac{2}{b}u + c,$$

where c is given above equation (2.26). It follows from (2.27) that the constant β is given by

$$\beta^2 = \frac{q(D-2)a^2\mu^2}{(\Delta + 2)\tilde{q} + 2}. \quad (2.29)$$

In terms of the original functions A , B and U appearing in the metric (2.1), the solution is given by

$$\begin{aligned}
e^{-\frac{\Delta'(D-2)}{2\tilde{q}}A} &= \frac{\lambda\sqrt{b}}{2\beta} e^{\frac{1}{2}\tilde{q}(\Delta+2)^c} \sinh(\beta t + \alpha) e^{(q\tilde{q} - \frac{b(D-2)}{2\tilde{q}})\mu t}, \\
e^{-\frac{b}{\Delta'+2}B} &= \frac{\lambda\sqrt{b}}{2\beta} \sinh(\beta t + \alpha) e^{(\frac{1}{2}\Delta - \frac{(q-1)b}{(D-2)(\Delta'+2)})^c} e^{\frac{qa^2}{\Delta'+2}\mu t}, \\
e^{-\frac{b}{2\epsilon a}\phi} &= \frac{\lambda\sqrt{b}}{2\beta} e^{\frac{1}{2}\tilde{q}(\Delta+2)^c} \sinh(\beta t + \alpha) e^{q\tilde{q}\mu t},
\end{aligned} \tag{2.30}$$

where $\Delta' = \Delta - 2(q-1)/(D-2)$.

It is of interest to examine the cosmological features of these solutions. Note that unlike the $\tilde{k} = 0$ case, there exists a limit where $\beta = 0 = \mu$. In this limit, all the functions A , B and ϕ depend linearly on $\log t$. When β and μ are non-vanishing, the time dependence of these functions is more complicated. Let us consider β and μ to be both positive. At the beginning, when $\sinh(\beta t + \alpha)$ vanishes, both the scale factors e^A and e^B diverge. As the coordinate t approaches infinity, e^A and e^B behave quite differently. For $\Delta > 0$, the scale factor e^B shrinks to zero, whilst the behaviour of the scale factor e^A depends on the sign of $\beta + \mu(q\tilde{q} - b(D-2)/2\tilde{q})$. This can be shown to be negative if $a^2\tilde{q}(\tilde{q}-1) > 2$. The scale factor e^A thus starts from infinity at the beginning and shrinks to a minimum and then grows indefinitely at late times. The behaviour of the dilaton depends on the sign of ϵ .

Case 3:

In this case, we take both k and \tilde{k} to be non-vanishing. As we mentioned earlier, the equations (2.14) cannot be solved in general. However there exists a special solution, where the first derivatives of the functions Z_1 , Z_2 and ϕ are proportional to one another. The equations (2.14) can be reduced to a single Liouville equation if we have

$$\begin{aligned}
Z_1 &= \frac{\Delta}{(D-2)a^2}u + \frac{(\Delta - \Delta\tilde{q} - \tilde{q})x_1 + \Delta\tilde{q}x_2 + 2\tilde{q}x_3}{2(D-2)a^2}, \\
Z_2 &= \frac{\Delta + 2 - 2q}{(D-2)a^2}u + \frac{\Delta qx_1 - \Delta(q-1)x_2 - 2(q-1)x_3}{2(D-2)a^2}, \\
-\frac{\phi}{\epsilon a} &= \frac{2(q-1)}{(D-2)a^2}u + \frac{q\tilde{q}x_1 - (q-1)\tilde{q}x_2 - (D-2)x_3}{(D-2)a^2},
\end{aligned} \tag{2.31}$$

where the constants x_1 , x_2 and x_3 are given by

$$e^{x_1} = \frac{\Delta}{(-k)(D-2)(q-1)a^2}, \quad e^{x_2} = \frac{\Delta + 2 - 2q}{(-\tilde{k})(D-2)(\tilde{q}-1)a^2}, \quad e^{x_3} = \frac{4(q-1)}{\lambda^2(D-2)a^2}. \tag{2.32}$$

The Liouville equation is given by $\ddot{u} - e^{2u} = 0$, and the first-order equation becomes $\dot{u}^2 - e^{2u} = 0$, with the solution $u = -\log t + c$. Here c is a constant of integration. Thus using the

freedom to choose c , we can relax one of the relations in (2.32). In this case, the behaviour of the metric components is

$$e^A \sim t^{-\frac{1}{D-2}}, \quad e^B \sim t^{-\frac{1}{D-2}}, \quad e^{\frac{\phi}{\epsilon a}} \sim t^{\frac{2(q-1)}{(D-2)a^2}}, \quad (2.33)$$

and so e^A and e^B decrease as t increases. For electric solutions ($\epsilon = 1$) the dilaton ϕ tends to infinity for large t , while for magnetic solutions ($\epsilon = -1$) the dilaton tends $-\infty$ as t increases. It is also not hard to express the scale factors in terms of the comoving time τ :

$$e^A \sim \tau, \quad e^B \sim \tau, \quad e^{\frac{\phi}{\epsilon a}} \sim \tau^{\frac{2(q-1)}{a^2}}. \quad (2.34)$$

3 Cosmological solutions with cosmological terms

In the previous section, we obtained cosmological solutions using n -form field strengths. Such solutions arise in massless supergravities in D -dimensions. Supergravities can also admit one or more cosmological terms, which can be viewed as the special case where $n = 0$. Such theories are in general massive, but they can nevertheless be obtained by the Kaluza-Klein reduction of massless supergravities in higher dimensions. For example, gauged massive supergravity in $D = 4$ [26] or $D = 7$ [27] can be obtained by the dimensional reduction of 11-dimensional supergravity [30] on a 7-sphere [28] or a 4-sphere [29]. The highest dimensional massive supergravity is the massive type IIA theory in $D = 10$ [31], which seems to indicate the possible existence of a 13-dimensional theory [32]. In fact, the number of massive supergravities in lower dimensions is far greater than that of massless supergravities, at least in the maximally supersymmetric case. It was shown recently that certain generalised Kaluza-Klein compactifications of the low-energy limits of string theory or M-theory can give rise to massive supergravities with cosmological terms in D -dimensions [33, 32, 34, 35]. The generalised reduction involves making an ansatz for a rank $(n - 1)$ potential such that its field strength contains a term that is a constant multiple of an harmonic n -form on the compactifying manifold. Such a phenomenon was also discussed earlier from a group theoretic point of view in [36]. In general, such massive supergravities contain cosmological terms. We are interested in cases where the bosonic Lagrangian can be consistently truncated to one containing just the metric, dilatonic scalar fields and cosmological terms, of the form:

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2}\sum_{\alpha}^N m_{\alpha}^2 e^{\vec{c}_{\alpha}\cdot\vec{\phi}}, \quad (3.1)$$

where $\vec{\phi} = (\phi_1, \phi_2, \dots)$ denotes a set of dilatonic scalar fields in the theory and \vec{c}_α are constant vectors. In the toroidal compactification, consistency of the truncation requires that the dilatonic vectors \vec{c}_α satisfy the following dot product relations [24]

$$M_{\alpha\beta} = 4\delta_{\alpha\beta} + \frac{2(D-1)}{D-2} . \quad (3.2)$$

In this case, the Lagrangian (3.1) can be further truncated to a single-scalar Lagrangian, of the form [24]

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2 e^{a\phi} , \quad (3.3)$$

where

$$\begin{aligned} a^2 &= \left(\sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \right)^{-1} , & \phi &= a \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \vec{c}_\alpha \cdot \vec{\phi} , \\ m_\alpha^2 &= a^2 m^2 \sum_{\beta} (M^{-1})_{\alpha\beta} = \frac{m^2}{N} . \end{aligned} \quad (3.4)$$

The constant a can be parameterised by writing $a^2 = \Delta + 2(D-1)/(D-2)$, with $\Delta = 4/N$. In some cases, the Lagrangian (3.3) can be embedded in a massive supergravity arising from the generalised reduction of the low-energy effective action of a string theory or M-theory on some other Ricci-flat manifold, such as K3, Calabi-Yau or a 7-dimensional Joyce manifold [34, 35]. The Lagrangian (3.1) gives rise to supersymmetric domain-wall solutions, which can be oxidised back to $D = 10$ or $D = 11$ where they become p -branes or intersecting p -branes. The situation for compactifications of string theory or M-theory where the internal manifold is not Ricci-flat is different. In particular, values of Δ other than $4/N$ can now arise in the resulting massive supergravity theories in the lower dimensions. For example, in gauged massive supergravity in $D = 7$, which can be obtained by compactifying 11-dimensional supergravity on a 4-sphere, the value of Δ for a certain cosmological term is -2 . The corresponding domain-wall solutions, although they can still be oxidised to $D = 11$ owing to the consistency of the reduction, will no longer become p -branes or intersecting p -branes. For example a special domain-wall solution, namely the AdS_7 spacetime solution of 7-dimensional gauged massive supergravity, becomes $\text{AdS}_7 \times S^4$ upon oxidation, and thus cannot be regarded as a p -brane in $D = 11$. In other words, in the generalised dimensional reduction discussed above, the harmonic function associated with the domain-wall solution in the lower dimension is precisely the same as the harmonic function associated with the higher-dimensional p -brane solution, implying that the lower-dimensional solution can be obtained from higher-dimensional one *via* the vertical dimensional reduction procedure [37, 38, 39, 40]. By contrast, in the spherical compactification the harmonic function in the

lower-dimensional domain-wall solution is not the same as the one in the higher-dimensional solution.

In this section, we are focussing on the construction of cosmological solutions that involve cosmological terms. Such solutions are magnetic, in the sense that they are the extrapolation to $n = 0$ of the magnetic ansatz discussed in section 2. Of course the cosmological terms can be dualised to give rise to D -form field strengths in D dimensions. In terms of the D -form field strengths, the corresponding solutions are instead electric. The cosmological solutions that we are going to discuss include both the magnetic solutions for cosmological terms and the electric solutions for D -form field strengths. It follows from (2.1) and the fact that $q = 0$ and $\tilde{q} = D - 1$ for both cases that the metric ansatz is given by

$$ds^2 = -e^{2(D-1)B} dt^2 + e^{2B} ds_{\tilde{q}}^2, \quad (3.5)$$

where we have chosen the gauge $U = (D - 1)B$. We shall first consider solutions from the single-scalar Lagrangian (3.3) (or its D -form dual with $e^{-a\phi} F_D^2$). The equations of motion are given by

$$\begin{aligned} \ddot{\phi} &= -\frac{1}{2}am^2 e^{2(D-1)Y+\Delta\phi/a}, & \ddot{Y} &= -\tilde{k}(D-2)e^{2(D-2)Y-2\phi/a}, \\ \frac{1}{2}m^2 e^{2(D-1)Y+\Delta\phi/a} - (D-1)(D-2)(\dot{Y} + \frac{\dot{\phi}}{(D-2)\epsilon a})^2 + \frac{1}{2}\dot{\phi}^2 & & & \\ &= \tilde{k}(D-1)(D-2)e^{2(D-1)Y-2\phi/a}. \end{aligned} \quad (3.6)$$

These equations can be solved in general when $\tilde{k} = 0$, giving a special case of the $\tilde{k} = 0$ solutions discussed in section 2. The solution is

$$\begin{aligned} e^{\frac{\Delta(D-2)}{2}B} &= \frac{m\sqrt{\Delta}}{2\beta} \cosh(\beta t + \alpha) e^{\frac{(D-2)a^2}{2}\mu t}, \\ e^{-\frac{\Delta\phi}{2a}} &= \frac{m\sqrt{\Delta}}{2\beta} \cosh(\beta t + \alpha) e^{(D-1)\mu t}, \end{aligned} \quad (3.7)$$

where the constants β and μ are related by

$$\beta^2 = \frac{1}{2}(D-1)(D-2)a^2\mu^2. \quad (3.8)$$

In this $\tilde{k} = 0$ case, the generalisation of the solution (3.7) to a multi-scalar solution with the dilaton vectors \vec{c}_α satisfying the dot products (3.2) is straightforward, and the solution is given by

$$\begin{aligned} e^{2(D-2)B} &= e^{\frac{2(D-2)a^2}{\Delta}\mu t} \prod_{\alpha=1}^N \left(\frac{m_\alpha}{\beta_\alpha} \cosh(\beta_\alpha t + \gamma_\alpha) \right), \\ e^{-\frac{1}{2}\vec{c}_\alpha \cdot \phi - (D-1)B} &= \frac{m_\alpha}{\beta_\alpha} \cosh(\beta_\alpha t + \gamma_\alpha), \end{aligned} \quad (3.9)$$

where $\Delta = 4/N$ and the constants β_a and μ are related by $\sum_\alpha \beta_\alpha^2 = 2(D-1)(D-2)a^2\mu^2/\Delta$.

Having obtained cosmological solutions involving cosmological terms, it is of interest to study their physical characteristics. For now, we shall consider them from the point of view of the D -dimensional theory itself. The Lagrangian (3.3) has a coupling constant given by $g_m = e^{-\frac{1}{2}a\phi}$. (The coupling constant for the dual theory with a D -form field strength is given by $g_e = 1/g_m$, since the dilaton reverses its sign under the dualisation.) In order to study the large $|t|$ behaviour of the coupling constant, we see from (3.7) that we need to compare the values of β and $(D-1)\mu$:

$$\beta^2 - (D-1)^2\mu^2 = \frac{1}{2}(D-1)(D-2)\Delta\mu^2. \quad (3.10)$$

Thus the behaviour of the coupling constant depends on the sign of the constant Δ . For positive Δ , the coupling constant g_e tends to zero at large $|t|$, whilst the coupling constant g_m tends to infinity. For cosmological terms in supergravity theories, the Δ values can sometimes be negative instead. For example, a pure cosmological term without any dilaton coupling has $\Delta = -2(D-1)/(D-2)$. (In this case, however, the discussion for the dilaton behaviour is irrelevant.) Another example is provided by gauged massive supergravities in $D = 4$ or $D = 7$, which arise from the spherical compactification of 11-dimensional supergravity. The value of Δ for the cosmological term associated with the gauge parameter is $\Delta = -2$. In this case, the coupling constant g_e runs from infinity to zero as the coordinate t goes from $-\infty$ to ∞ , and the coupling constant g_m behaves in exactly the opposite way (here without loss of generality we assume that μ is positive). Similarly, we can discuss the behaviour at large $|t|$ for the scale factor $R = e^B$. In this case, we need to compare the value of β with $a^2(D-2)/2\mu$:

$$\beta^2 - (\frac{1}{2}(D-2)a^2\mu)^2 = -\frac{1}{4}(D-2)^2a^2\Delta\mu^2. \quad (3.11)$$

Again the behaviour of the scale factor R depends only on the sign of Δ . If Δ is negative, the scale factor diverges at large $|t|$. On the other hand, if Δ is positive, the scale factor runs from zero to infinity as the t coordinate runs from ∞ to $-\infty$.

In order to discuss the evolution of the solutions, it is useful to introduce the comoving time coordinate τ , defined by $ds^2 = -d\tau^2 + e^{2B}ds_{\hat{q}}^2$, which implies that $\tau = \int^t e^{(D-1)B}dt$. Thus at large t , we have $\tau \sim \exp((D-1)(\beta|t| - \frac{1}{2}(D-2)a^2t)/(2(D-2)))$. If Δ is positive, the universe starts at finite τ_0 when the scale factor is zero, and then expands to infinity as τ goes to infinity. In this case, since we have $R = \tau^{1/(D-1)}$, the expansion of the universe slows down at large τ , but there is no inflation in the early stages. If Δ is negative, the

situation is different. In this case, the comoving time τ runs from infinity to a finite value and then to infinity again as t goes from $-\infty$ to ∞ . Thus we can also consider that the universe starts at a finite τ_0 when its size is minimal, and then it expands to infinity as τ goes to ∞ .

The physical properties of the cosmological solutions with cosmological terms were discussed above in the context in the D -dimensional theories themselves. Since these theories are dimensional reductions of more fundamental theories such as string theory or M-theory, their cosmological features can be more appropriately analysed in ten or eleven dimensions. As we mentioned earlier, there are numerous massive supergravity theories in lower dimensions, and we shall give only a few examples to illustrate the oxidation of the lower-dimensional solutions and to discuss their higher-dimensional characteristics.

One simple example is the electrically-charged cosmological solution in $D = 4$ using the 4-form coming from the 4-form field strength in M-theory in $D = 11$. The 4-dimensional solution is given by

$$\begin{aligned} e^{4B} &= \frac{m}{\beta} \cosh(\beta t + \alpha) e^{7\mu t} , \\ e^{-2\phi/\sqrt{7}} &= \frac{m}{\beta} \cosh(\beta t + \alpha) e^{3\mu t} , \end{aligned} \quad (3.12)$$

where $\beta = \sqrt{21}\mu > 0$. The solution can be oxidised to $D = 11$, giving

$$\begin{aligned} ds_{11}^2 &= e^{\frac{\sqrt{7}}{3}\phi} ds_4^2 + e^{-\frac{2}{3\sqrt{7}}\phi} ds_7^2 \\ &= -e^{6B+\frac{\sqrt{7}}{3}\phi} dt^2 + e^{2B+\frac{\sqrt{7}}{3}\phi} ds_3^2 + e^{-\frac{2}{3\sqrt{7}}\phi} ds_7^2 . \end{aligned} \quad (3.13)$$

In this 11-dimensional metric, we see that the scale factor for the 3-dimensional space $R = e^{B+\phi\sqrt{7}/6} \sim (\cosh(\beta t + \alpha))^{-1/2}$ now shrinks to zero at large $|t|$. On the other hand, the scale factor $R' = e^{-\phi/(3\sqrt{7})}$ for the 7-dimensional internal space runs from infinity to a minimum size and then to infinity again as the coordinate t runs from $-\infty$ to ∞ . Note that the comoving time coordinate τ runs from 0 to τ_0 , where R' reaches its minimum, and then goes to infinity. Thus the 11-dimensional metric effectively describes an 8-dimensional expanding universe which starts at time τ_0 , where it has a Planckian size. Note that the time interval $(0, \tau_0)$ is of the Planckian scale too. Since the scale factor R' has only one minimum, at $\tau = \tau_0$, it implies that $\dot{R}' \equiv dR'/d\tau > 0$ for all times $\tau_0 < \tau < \infty$. It also implies that close to the beginning of the universe, $\tau \sim \tau_0$, we have $\ddot{R}' > 0$ and hence an inflationary expansion. On the other hand as $\tau \rightarrow \infty$ we have $R' \sim \tau^{(\beta+3\mu)/(\beta+21\mu)}$, and hence $\ddot{R}' < 0$, implying a slowing down of the expansion rate of the universe.

Now let us consider a 4-dimensional magnetic cosmological solution in $D = 4$ using a cosmological term. The 4-dimensional theory in question is obtained [35] from a generalised compactification of M-theory on a 7-torus or on a 7-dimensional Joyce manifold [41]. The solution involves 7 cosmological terms, and hence has $a^2 = 25/7$ and $\Delta = 4/7$ (see [35] for the configuration of the cosmological terms). We consider only the case where all the cosmological constants are equal, for which the solution is given by

$$\begin{aligned} e^{\frac{4}{7}B} &= \frac{m}{\sqrt{7}\beta} \cosh(\beta t + \alpha) e^{\frac{25}{7}\mu t} , \\ e^{-\frac{2}{5\sqrt{7}}\phi} &= \frac{m}{\sqrt{7}\beta} \cosh(\beta t + \alpha) e^{3\mu t} , \end{aligned} \quad (3.14)$$

where $\beta = 5\sqrt{3/7}\mu$. Again this solution can be oxidised to $D = 11$, in fact giving the same result (3.13) as in the electric solution described above, except that now B and ϕ are given by (3.14) rather than (3.12). In this case, one can easily verify that R shrinks and R' expands, effectively describing an 8-dimensional expanding universe. As in the example above, at the beginning of the universe $\tau = \tau_0$ there is an inflationary period, whilst at large times the expansion rate slows down.

An interesting massive supergravity theory is the one in $D = 7$ with a topological mass term, which can be obtained by a generalised compactification of M-theory on a 4-torus or a K3 manifold. Here, the generalised Kaluza-Klein ansatz implies that the 4-form field strength has an extra term that is proportional to the volume form of the internal space. The cosmological constant in $D = 7$ is the square of this constant of proportionality. The 7-dimensional solution takes the form

$$\begin{aligned} e^{10B} &= \frac{m}{\beta} \cosh(\beta t + \alpha) e^{16\mu t} , \\ e^{-\frac{1}{2}\sqrt{\frac{5}{2}}\phi} &= \frac{m}{\beta} \cosh(\beta t + \alpha) e^{6\mu t} , \end{aligned} \quad (3.15)$$

where $\beta = 4\sqrt{6}\mu$. It can be oxidised to the $D = 11$ solution

$$\begin{aligned} ds_{11}^2 &= e^{\frac{2}{3}\sqrt{\frac{2}{5}}\phi} ds_7^2 + e^{-\frac{1}{3}\sqrt{\frac{5}{2}}\phi} ds_4^2 \\ &= -e^{12B+\frac{2}{3}\sqrt{\frac{2}{5}}\phi} dt^2 + e^{2B+\frac{2}{3}\sqrt{\frac{2}{5}}\phi} ds_6^2 + e^{-\frac{1}{3}\sqrt{\frac{5}{2}}\phi} ds_4^2 , \end{aligned} \quad (3.16)$$

It is easy to see that now the scale factor for the six-dimensional space with metric ds_6^2 is $R = e^{B+\frac{1}{3}\sqrt{\frac{2}{5}}\phi} \sim (\cosh(\beta t + \alpha))^{-1/6}$, implying that the space shrinks to zero at large $|t|$, whilst the scale factor $R' = e^{-\frac{1}{6}\sqrt{\frac{5}{2}}\phi}$ diverges at large $|t|$. Thus the solution effectively describes a 5-dimensional expanding universe. The comoving time coordinate runs from

zero to a Plankian time τ_0 , where the scale factor R' is a minimum, and then runs to infinity as the 4-dimensional space expands while the 6-dimensional space contracts. Since at the starting point of the universe $\tau = \tau_0$ the scale factor R' is a minimum, we have that $\ddot{R}' > 0$ in the early universe. At large τ , we have $R' \sim \tau^{(\beta+6\mu)/(\beta+24\mu)}$, implying that $\dot{R}' > 0$ but $\ddot{R}' < 0$.

We can alternatively oxidise the solution to $D = 10$ instead. In this case, it will effectively describe a 4-dimensional expanding universe. To see this, we note that the metric in $D = 10$ is given by

$$\begin{aligned} ds_{10}^2 &= e^{\frac{9}{8\sqrt{10}}\phi} ds_7^2 + e^{-\frac{3}{8}\sqrt{\frac{5}{2}}\phi} ds_3^2 \\ &= -e^{12B+\frac{9}{8\sqrt{10}}\phi} dt^2 + e^{2B+\frac{9}{8\sqrt{10}}\phi} ds_6^2 + e^{-\frac{3}{8}\sqrt{\frac{5}{2}}\phi} ds_3^2. \end{aligned} \quad (3.17)$$

Again the scale factor for the six-dimensional space with metric ds_6^2 shrinks whilst the 3-dimensional space with metric ds_3^2 expands, effectively describing a 4-dimensional expanding universe, with an inflationary period at early times, and a slowing down of the expansion rate these days. In fact this is a solution in $D = 10$ which is supported by a 3-form field strength. General cosmological solutions involving 3-form field strengths in $D = 10$ were also discussed in [1, 19]. It is interesting to study the behaviour of the string coupling in $D = 10$. There are two maximally supersymmetric string theories in $D = 10$, namely the type IIA and type IIB strings. If we oxidise the $D = 7$ solution to $D = 10$, where the $D = 7$ theory is taken to be embedded in the type IIA theory, we find that the corresponding string coupling constant is given by $g = e^{-\sqrt{5/2}\phi/4}$, which diverges at large $|t|$. If on the other hand we view the $D = 7$ theory as being embedded in the type IIB string, and oxidise it to $D = 10$, there are two possible outcomes. If the charge is carried by the NS-NS 3-form, then the coupling constant is the same of the type IIA case, and diverges at large $|t|$; on the other hand, if the charge is carried by the R-R 3-form, then the coupling constant is the inverse of that in the type IIA case, and hence goes to zero at large $|t|$.

4 Relation between cosmological and p -brane solutions

It has been observed recently that there is a similarity in certain cases between the cosmological solutions of the kind we are discussing in this paper, and type 1 black p -brane solutions in string theory [1]. In this section, we shall explore this in a more general setting, and show that there is an exact one-to-one map between the $k = 1$, $\tilde{k} = 0$ type 1 cosmological solutions and the type 1 black p -brane solutions that were discussed in [4]. In

the special case when $p = 0$, these latter solutions are the same as standard non-extremal black holes. Thus a subset of the type 1 cosmological solutions can be mapped into standard black-hole solutions. However, when p is greater than zero, the type 1 black p -brane solutions obtained in [4] are not ordinary type 2 [6] black p -branes. Thus in cases where the cosmological solutions map into p -brane solutions with $p > 0$, it is to these non-standard type 1 black p -branes, rather than the standard type 2 ones, that the mapping occurs.

To see this in detail, we begin by recalling the structure of the type 1 black p -branes constructed in [4]. They have metrics which take the form

$$ds^2 = e^{2\tilde{A}} (-d\tilde{t}^2 + dx^i dx^i) + e^{2\tilde{B}} (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2) , \quad (4.1)$$

where \tilde{A} and \tilde{B} are functions only of \tilde{r} . Here x^i are the p spatial coordinates on the world-volume of the p -brane, and $d\Omega^2$ is the metric on the unit $(\tilde{d}+1)$ -sphere, where $\tilde{d} = D - d - 2$ and $d = p + 1$. The metric (4.1) is of the form of the standard ansatz for extremal p -branes, except that there the further restriction $d\tilde{A} + \tilde{d}\tilde{B} = 0$ is imposed. In [4], more general solutions, which are non-extremal, were obtained by relaxing this assumption, so that there is an additional variable $\tilde{X} \equiv d\tilde{A} + \tilde{d}\tilde{B}$ appearing in the equations of motion. This has the solution $e^{\tilde{X}} = 1 - \tilde{\kappa}^2 \rho^2$, where $\tilde{\kappa}$ is a constant and $\rho \equiv \tilde{r}^{-\tilde{d}}$. It is natural to introduce a further redefined radial coordinate ξ , such that $e^{\tilde{X}} \partial/\partial \rho = \partial/\partial \xi$, and thus $\tilde{\kappa} \rho = \tanh \tilde{\kappa} \xi$. The general type 1 black p -brane solutions take the form [4]

$$\begin{aligned} e^{-\frac{(D-2)\Delta}{2\tilde{d}}\tilde{A}} &= \frac{\tilde{\lambda}\sqrt{\Delta}}{2\tilde{d}\tilde{\beta}} \sinh(\tilde{\beta}\xi + \alpha) e^{a^2(D-2)\tilde{\mu}\xi/(2\tilde{d})} , \\ e^{\frac{(D-2)\Delta}{2\tilde{d}}\tilde{B}} &= \frac{\tilde{\lambda}\sqrt{\Delta}}{2\tilde{d}\tilde{\beta}} \sinh(\tilde{\beta}\xi + \alpha) e^{a^2(D-2)\tilde{\mu}\xi/(2\tilde{d})} (\cosh \tilde{\kappa}\xi)^{-\frac{(D-2)\Delta}{d\tilde{d}}} , \\ e^{\frac{\Delta}{2ac}\phi} &= \frac{\tilde{\lambda}}{2\tilde{d}\tilde{\beta}} \sinh(\tilde{\beta}\xi + \alpha) e^{-d\tilde{\mu}\xi} . \end{aligned} \quad (4.2)$$

The various constants of integration are subject to the constraint $\tilde{d}\tilde{\beta}^2 = 2(\tilde{d}+1)\Delta\tilde{\kappa}^2 - \frac{1}{2}a^2d(D-2)\tilde{\mu}^2$.

The metrics described by these solutions have an outer event horizon where the function $e^{\tilde{X}} = 1 - \tilde{\kappa}^2 \rho^2$ vanishes. In the case when $p = 0$, they describe precisely the standard non-extremal black-hole solutions of string theory, with $\tilde{\kappa}$ being a non-extremality parameter that vanishes in the extremal limit. However, when p is greater than zero, (4.1) is quite different from the metric of a standard type 2 black p -brane, for which one would have [6]

$$ds^2 = e^{2\hat{A}} (-e^{2f} dt^2 + dx^i dx^i) + e^{2\hat{B}} (e^{-2f} dr^2 + r^2 d\Omega^2) , \quad (4.3)$$

with $e^{2f} = 1 - \kappa r^{-\tilde{d}}$ and

$$\begin{aligned} e^{-\frac{(D-2)\Delta}{2\tilde{d}}\hat{A}} &= 1 + \frac{\kappa}{r^{\tilde{d}}} \sinh^2 \mu , \\ e^{\frac{(D-2)\Delta}{2\tilde{d}}\hat{B}} &= 1 + \frac{\kappa}{r^{\tilde{d}}} \sinh^2 \mu , \\ e^{\frac{\Delta}{2a\epsilon}\phi} &= 1 + \frac{\kappa}{r^{\tilde{d}}} \sinh^2 \mu . \end{aligned} \quad (4.4)$$

These solutions also have an outer horizon where $e^{2f} = 1 - \kappa r^{-\tilde{d}}$ vanishes. However, it is clear that the structure of these metrics is not the same as those of (4.1) when $p > 0$. In particular, in (4.1) the sign of the spatial p -brane metric $dx^i dx^i$, as well as the dt^2 part, also reverses when the horizon is crossed, whereas in the standard metrics (4.3) only the dt^2 part of the p -brane world-volume metric reverses sign. The two types of solution become equivalent if $p = 0$.

It is clear that the solutions (4.2) have a very similar structure to the cosmological solutions given in section 2. In fact, as we shall show below, the type 1 p -brane solutions (4.1) and (4.2) can be mapped into the type 1 cosmological solutions (2.23) with $k = 1$ and $\tilde{k} = 0$. Under this mapping, the q -dimensional space in the cosmological solution, which is flat since $\tilde{k} = 0$, becomes the world volume of the p -brane, with $p = \tilde{q} - 1$. Depending on whether the function e^{2B} in (2.1) is positive or negative after the mapping, either one or $(\tilde{q} - 1)$ of these Euclidean coordinates must be Wick rotated. The coordinate t , which becomes spacelike either through a Wick rotation or because e^{2U} becomes negative, together with the coordinates of the q -dimensional space in (2.1), become the coordinates of the transverse space of the p -brane solution.

To see this, it is instructive first to write the metric (2.1) for $k = 1$ and $\tilde{k} = 0$ in the following form:

$$ds^2 = -e^{2U} dt^2 + e^{2A} d\tilde{s}_q^2 + e^{2B} (dy^2 + dy^j dy^j) , \quad (4.5)$$

where j runs over $\tilde{q} - 1$ values. It is clear that the p -brane solutions can be mapped to the cosmological ones if the following relations hold:

$$e^{2\tilde{B}} dr^2 = -e^{2U} dt^2, \quad e^{2\tilde{B}} r^2 = e^{2A}, \quad e^{2\tilde{A}} = (-1)^\delta e^{2B} , \quad (4.6)$$

where δ takes the value 0 or 1. Furthermore, we need to identify $\tilde{t} = iy$, $x^i = y^j$ if $\delta = 0$ or $\tilde{t} = y$, $x^j = iy^j$ if $\delta = 1$, implying $d = \tilde{q}$ and $\tilde{d} = q - 1$. From the above relations, we immediately see that the p -brane coordinate ξ and the constant κ are related to the time coordinate t and the constant c of the cosmological solutions by

$$\xi = i^{-\tilde{q}\delta+1} (q-1)t + \nu, \quad \tilde{\kappa} = \frac{i^{-\tilde{q}\delta+1}}{2(q-1)} c , \quad (4.7)$$

where ν is a constant. Now, using these relations in the explicit forms of A, B, \tilde{A} and \tilde{B} , it is easy to relate the other parameters that appear in the p -brane and cosmological solutions:

$$\begin{aligned}\tilde{\beta} &= \frac{i^{-\delta\tilde{q}+1}}{(q-1)}\beta, & \tilde{\mu} &= \frac{i^{-\delta\tilde{q}+1}}{(q-1)}\mu, \\ \tilde{\lambda} &= i^{-\frac{a^2(D-2)\delta}{2(q-1)}+1} e^{-\frac{a^2(D-2)\tilde{\mu}\nu}{2(q-1)}} \lambda, & \alpha &= \gamma - i\frac{\pi}{2}.\end{aligned}\tag{4.8}$$

The above demonstration illustrates that for the case $k = 1$ and $\tilde{k} = 0$, the type 1 cosmological solutions are identical to those of the type 1 non-extremal p -brane solitons obtained in [4], up to general coordinate transformations that may include Wick rotations. In fact once the equivalence of the type 1 p -brane ansatz (4.1) and the cosmological ansatz (4.5) for $k = 1, \tilde{k} = 0$ metrics is established the equivalence of the solutions is guaranteed, since in each case the most general solutions compatible with the ansatz were obtained. Although the non-extremal p -branes are not BPS saturated, and thus can suffer modifications at the quantum level, they do, of course, have extremal limits in which they become BPS saturated. This occurs when the parameter $\tilde{\kappa}$ goes to zero. It is of interest to enquire what happens to the corresponding cosmological solutions in this limit. From the constraint given under (4.2), we see that setting $\tilde{\kappa} = 0$ in the p -brane solutions implies that the constants $\tilde{\beta}$ and $\tilde{\mu}$ must go to zero, and thus from (4.8) we must have β going to zero also. From (2.20), this would imply that Φ is either singular, or complex. Thus there is no limit possible in which the cosmological solutions can become supersymmetric. (The fact that they are non-supersymmetric was observed in [1].) It is interesting, however, that the near-extremal regime of non-extremal p -branes does map into real cosmological solutions. Thus it may be that some of the desirable properties associated with near-extremality for p -branes may have their analogues in cosmological string solutions.

So far we have seen that the single-scalar purely electric or purely magnetic type 1 cosmological solutions can be one-to-one mapped to the associated type 1 non-extremal p -branes. The generalisation of this mapping to multi-scalar or dyonic solutions is straightforward. The corresponding non-extremal p -brane and cosmological solutions can be found in [4] and [19] respectively.

As we discussed in section 2, cosmological solutions also exist for other values of k and \tilde{k} . As shown above when $\tilde{k} = 0$, the corresponding flat space with metric ds_q^2 can be reinterpreted as the world volume of the p -brane. It seems that such an interpretation breaks down when $\tilde{k} \neq 0$, and such cosmological solutions would map into solutions that do not have an interpretation as p -branes. On the other hand, there is no such restriction on k , which defines the structure of the transverse space of the corresponding p -brane. When

$k = 1$, we have seen that the cosmological solutions can be mapped to isotropic p -branes. By the same token, we can obtain new non-extremal p -brane solutions by mapping the cosmological solutions with other values of k . In the case of $k = 0$, the corresponding p -brane solutions allow extremal limits, and were obtained in [42]. These supersymmetric solutions describe configurations of p -branes uniformly distributed over a q -dimensional hyperplane. They become domain walls after compactifying these q coordinates [34].

So far, we related type 1 cosmological solutions and the type 1 non-extremal p -branes. As was shown in [5, 6], there is another universal way to blacken extremal p -branes, namely the type 2 p -branes given by (4.3) and (4.4). It was observed in [2] that in the region where e^{2f} is negative, the signature of the t and r coordinates reverses, leading to a natural interpretation of the interior p -brane solution, inside the horizon at $e^{2f} = 0$, as a cosmological solution. In this case there is no need to perform any Wick rotation, since the sign reversal of the function e^{2f} automatically gives the original r coordinate a timelike interpretation, while the original time coordinate becomes spatial. For the case $p = 0$, as we have already remarked, the standard and the non-standard black p -branes coincide, and the description of the mapping to the cosmological solutions is identical. Let us therefore consider the remaining cases, when $p > 0$. It seems to be natural to consider the universe as beginning at $r = 0$, since this is a point where the curvature is singular. The universe will then evolve to the horizon at $r = \kappa^{1/\tilde{d}}$. At $r = 0$ the scale factors for $dx_i dx_i$ and for $d\Omega^2$ are zero, expanding to a finite size at the horizon. In this interval, the comoving time $\tau = \int^r e^{B-f} dr$ covers a finite range. The scale factor for dt^2 at the horizon is zero, whilst it is proportional to $r^{\tilde{d}(2a^2/d-1)}$ at small r . Thus to give a satisfactory cosmological model we must have $a^2 \geq d/2$, since there should not be spatial directions that grow to infinite size at a finite comoving time.

To summarise, we have seen that there are two types of non-extremal p -branes solutions: the non-standard (type 1) with the metric (4.1), and the standard (type 2) with the metric (4.3). Correspondingly there are type 1 and type 2 cosmological solutions. We observed that the type 1 black p -branes are identical to the type 1 cosmological solutions that were obtained in section 2, with $k = 1$ and $\tilde{k} = 0$. In general, the mapping between these p -brane metrics and cosmological metrics requires the Wick rotation of one or more coordinates. It is also straightforward to observe that the interior of the type 2 black p -branes can be interpreted as type 2 cosmological models. In this case, no Wick rotation of coordinates is necessary. It is of interest to study the inter-relationship of these solutions

Since the time coordinate t of a type 2 black p -brane becomes spacelike inside the

horizon, while the original r coordinate becomes timelike, one can choose to perform a Kaluza-Klein reduction of the interior metric in which t is compactified. This gives rise to a lower-dimensional cosmological solution with the general form of the type 1 models. There is however a slight complication in general, since the lower-dimensional solution will acquire an additional scalar degree of freedom. However, if we start with a higher-dimensional type 2 solution with no dilaton excitation, its compactification gives precisely a type 1 cosmological solution. Since this solution can be mapped to a type 1 black p -brane by Wick rotation, it implies that under appropriate circumstances type 1 black p -branes can also be obtained from type 2 black p -branes by compactifying the time coordinate, made spacelike by virtue of the Wick rotation. We shall now illustrate this, taking the type 2 black membrane in $D = 11$ as an example. Its metric is given by

$$ds_{11}^2 = H^{-\frac{2}{3}} (-e^{2f} dt^2 + dx^i dx^i) + H^{\frac{1}{3}} (e^{-2f} dr^2 + r^2 d\Omega_7^2) , \quad (4.9)$$

where $e^{2f} = 1 - \kappa r^{-6}$ and $H = 1 + \kappa r^{-6} \sinh^2 \mu$. If we simply compactify the solution on one of the world-volume spatial coordinates x^i , it reduces to a type 2 black string in $D = 10$. However we can instead perform Wick rotations to make the coordinate t spacelike, and the metric $dx^i dx^i$ to be Minkowskian, and then compactify the Wick rotated t coordinate to get a different dimensionally-reduced 10-dimensional metric ds_{10}^2 , given by $ds_{11}^2 = e^{\varphi/6} ds_{10}^2 + e^{-4\varphi/3} d(it)^2$. Thus we find

$$\begin{aligned} ds_{10}^2 &= H^{-\frac{3}{4}} e^{\frac{1}{4}f} dx^i dx^i + H^{\frac{1}{4}} e^{\frac{1}{4}f} (e^{-2f} dr^2 + r^2 d\Omega_7^2) , \\ e^{2\varphi} &= H e^{-3f} . \end{aligned} \quad (4.10)$$

This clearly has the structure of a type 1 black string solution, in that the metric coefficients for the time direction (one of the two x^i coordinates) and the spatial world-sheet direction are identical. Indeed, it is not difficult to show that (4.10) is precisely of the form of the type 1 black metrics (4.1), with

$$\tilde{\kappa} = 4\kappa , \quad \tilde{\mu} = -2\kappa , \quad r = \tilde{r} \left(1 + \frac{\kappa}{\tilde{r}^6} \right)^{\frac{1}{3}} . \quad (4.11)$$

Thus we see that the dimensional reduction of the Wick-rotated time coordinate of a dilaton-free type 2 black p -brane gives a special case of a type 1 black $(p-1)$ -brane in one lower dimension.

A special case of the type 2 \rightarrow type 1 reductions discussed above is when one begins with a black hole in the higher dimension. In order to make an exact correspondence with the type 1 solutions, we should again consider examples where there is no dilaton involved

in the higher-dimensional solution. A simple example is provided by considering black-hole solutions in the $N = 1$ string in six dimensions, where the charge is carried by a Yang-Mills field. Including the effects of loop corrections, it was shown in [43] that the relevant part of the effective Lagrangian is given by

$$\mathcal{L} = eR - \frac{1}{2}e (\partial\phi)^2 - \frac{1}{4}(ve^{-\phi/\sqrt{2}} + \tilde{v}e^{\phi/\sqrt{2}}) F^2 , \quad (4.12)$$

where v and \tilde{v} are constants. This admits a black-hole solution [44] where ϕ is a constant, given by $e^{\phi\sqrt{2}} = v/\tilde{v}$, and the metric is given by

$$ds_6^2 = -H^{-2} e^{2f} dt^2 + H^{\frac{2}{3}} (e^{-2f} dr^2 + r^2 d\Omega_4^2) , \quad (4.13)$$

where $H = 1 + \kappa r^{-3} \sinh^2 \mu$ and $e^{2f} = 1 - \kappa r^{-3}$. Performing a dimensional reduction of the t coordinate (having either first Wick-rotated it, or else by considering the interior solution where e^{2f} is negative), we obtain the 5-dimensional solution

$$\begin{aligned} ds_5^2 &= e^{\frac{2}{3}f} (e^{-2f} dr^2 + r^2 d\Omega_4^2) , \\ e^{-\sqrt{\frac{3}{2}}\varphi} &= H^{-2} e^{2f} . \end{aligned} \quad (4.14)$$

In the exterior region $e^{2f} > 0$ this describes a non-extremal instanton with a Euclidean signature; in the interior region it can be viewed as a cosmological solution, since the coordinate r becomes timelike. Note that the metric is independent of the electric charge, which appears only in the harmonic function H . In fact this is a general phenomenon that will occur whenever a black-hole solution is compactified in the time direction. Thus starting from the black-hole metric

$$ds_D^2 = -H^{-\frac{4(D-3)}{\Delta(D-2)}} e^{2f} dt^2 + H^{4\Delta(D-2)} (e^{-2f} dr^2 + r^2 d\Omega_{D-2}^2) , \quad (4.15)$$

we reduce it to $(D-1)$ dimensions by applying the standard Kaluza-Klein reduction on the time coordinate, $ds_D^2 = e^{2\alpha\varphi} ds_{D-1}^2 - e^{-2(D-3)\alpha\varphi} dt^2$, implying that

$$\begin{aligned} ds_{D-1}^2 &= e^{\frac{2f}{D-3}} (e^{-2f} dr^2 + r^2 d\Omega_{D-2}^2) , \\ e^{\alpha\varphi} &= H^{\frac{2}{\Delta(D-2)}} e^{-\frac{f}{D-3}} , \end{aligned} \quad (4.16)$$

where $\alpha^{-2} = 2(D-2)(D-3)$.

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